Functions of Several Variables (CC-09)

Partial derivatives of second order :-

The second order partial derivatives of f are denoted by

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{xy}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{yx}$$

The second order partial derivatives of a particular point (a,b) are denoted by.

$$\left[\frac{\partial^2 f}{\partial x^2}\right]_{(a,b)}, \quad \frac{\partial^2 f}{\partial x^2}(a,b), \quad f_{xx}(a,b)$$

$$\left[\frac{\partial^2 f}{\partial y^2}\right]_{(a,b)}, \quad \frac{\partial^2 f}{\partial y^2}(a,b), \quad f_{yy}(a,b) \text{ and so on.}$$

Thus,

$$f_{xx}(a,b) = \lim_{h\to 0} \frac{f_x(a+h,b) - f_x(a,b)}{h}$$

$$f_{xy}(a,b) = \lim_{h\to 0} \frac{f_y(a+h,b) - f_y(a,b)}{h}$$

$$f_{yx}(a,b) = \lim_{k\to 0} \frac{f_x(a,b+k) - f_x(a,b)}{k}$$

$$f_{yy}(a,b) = \lim_{k\to 0} \frac{f_y(a,b+k) - f_y(a,b)}{k}$$

in case the limit exists.

Ex1. Prove that $\frac{\partial^2 f(0,0)}{\partial x \partial y} \neq \frac{\partial^2 f(0,0)}{\partial y \partial x}$ for the function

$$f(x,y) = xy\frac{x^2-y^2}{x^2+y^2}, \quad x^2+y^2 \neq 0$$
$$= 0 \quad , x^2+y^2 = 0$$

Sol^n:- We have,

$$f_{xy}(x,y) = \lim_{h\to 0} \frac{f_y(x+h,y) - f_y(x,y)}{h}$$

$$\therefore f_{xy}(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h} \quad \text{——} (1)$$

Now,

$$f_y(x,y) = \lim_{k\to 0} \frac{f(x,y+k) - f(x,y)}{k}$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0 \quad (\because k \neq 0)$$

$$f_y(h,0) = \lim_{k\to 0} \frac{f(h,k) - f(h,0)}{k} = \lim_{k\to 0} \frac{hk\frac{h^2-k^2}{h^2+k^2} - 0}{k}$$
$$= h.$$

From (1) we get,

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{h-0}{h} = \lim_{h\to 0}\frac{h}{h} = 1 \quad (\because h \neq 0)$$

Again,

$$f_{yx}(x,y) = \lim_{k\to 0} \frac{f_x(x,y+k) - f_x(x,y)}{k}$$

$$\therefore f_{yx}(0,0) = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k} \quad \text{——} (2)$$

Now,

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$\therefore f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0 \quad (\because h \neq 0)$$

$$f_x(0,k) = \lim_{h\to 0} \frac{f(h,k) - f(0,k)}{h}$$
$$= \lim_{h\to 0} \frac{hk\frac{h^2-k^2}{h^2+k^2} - 0}{h} = -k$$

From (2) we get,

$$f_{yx}(0,0) = \lim_{k\to 0} \frac{-k-0}{k} = -1 \quad (\because k \neq 0)$$

$$\therefore f_{xy}(0,0) \neq f_{yx}(0,0).$$

Ex2. If $f(x,y) = xy$, when $|y| \leq |x|$
$= -xy$, when $|y| > |x|$

Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

Schwarz's Theorem :- If f_y exists in a certain nbd of a point (a,b) of the domain of definition of a function f and f_{yx} is continuous at (a,b), then $f_{xy}(a,b)$ exists and is equal to $f_{yx}(a,b)$.

Ex. If $f(x,y) = \frac{x^2y^2}{x^2+y^2}$ if $x^2+y^2 \neq 0$

$= 0$ if $x^2+y^2=0$.

Show that $f_{xy}(0,0) = f_{yx}(0,0)$; but f does not satisfy Schwarz theorem.

Sol^n:- We see that

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = 0$$

For $(x,y) \neq (0,0)$

$$f_x(x,y) = \frac{2xy^4}{(x^2+y^2)^2}, \quad f_y(x,y) = \frac{2x^4y}{(x^2+y^2)^2}$$

$$\therefore f_{yx}(0,0) = \lim_{k\to 0} \frac{f_x(0,k)-f_x(0,0)}{k} = 0$$

and $$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_y(h,0)-f_y(0,0)}{h} = 0$$

so that $f_{xy}(0,0) = f_{yx}(0,0)$.

Now see that, $$f_{yx}(x,y) = \frac{8xy^3(x^2+y^2) - 2xy^4 \cdot 2(x^2+y^2) \cdot 2y}{(x^2+y^2)^4}$$

$$= \frac{8x^3y^3}{(x^2+y^2)^3}$$

Let $(x,y) \to (0,0)$ along the line $y = mx$, where $-\infty < m < \infty$

$$\therefore \lim_{\substack{x\to 0\\ y\to 0}} f_{yx}(x,y) = \lim_{x\to 0} \frac{8x^3 \cdot m^3x^3}{(x^2+m^2x^2)^3} = \frac{8m^3}{(1+m^2)^3}$$

which is different for different values of m.

Hence $\lim_{\substack{x\to 0\\ y\to 0}} f_{yx}(x,y)$ does not exists.

Consequently $f_{yx}(x,y)$ is not continuous at $(0,0)$.

Thus f does not satisfy Schwarz theorem.

Ex1. If $u = \frac{y}{z} + \frac{z}{x} + \frac{x}{y}$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

Solⁿ:- Differentiating the given expression partially w.r.t. x we get,

$$\frac{\partial u}{\partial x} = -\frac{z}{x^2} + \frac{1}{y} \quad \text{i.e.} \quad x\frac{\partial u}{\partial x} = -\frac{z}{x} + \frac{x}{y}$$

Similarly, $$y\frac{\partial u}{\partial y} = -\frac{x}{y} + \frac{y}{z}$$

and $$z\frac{\partial u}{\partial z} = -\frac{y}{z} + \frac{z}{x}$$

Now,
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -\frac{z}{x} + \frac{x}{y} - \frac{x}{y} + \frac{y}{z} - \frac{y}{z} + \frac{z}{x}$$
$$= 0 \text{ (Proved)}$$

Ex2. If $u = \sin\frac{x}{y} + \tan\frac{y}{x}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

Solⁿ:- H.W.

Ex3. If $f(x,y,z) = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$, then show that

$$f_x + f_y + f_z = 0.$$

Solⁿ:- Here $f(x,y,z) = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x & y & z \end{vmatrix}$

$$= \begin{vmatrix} 1 & 0 & 0 \\ x^2 & y^2-x^2 & z^2-x^2 \\ x & y-x & z-x \end{vmatrix}$$

$$= \begin{vmatrix} y^2-x^2 & z^2-x^2 \\ y-x & z-x \end{vmatrix}$$

$$= (y-x)(z-x)\begin{vmatrix} y+x & z+x \\ 1 & 1 \end{vmatrix}$$

$$= (y-x)(z-x)(y+\cancel{x}-z-\cancel{x})$$

i.e. $f(x,y,z) = (x-y)(x-z)(y-z)$

$\therefore \log f = \log(x-y) + \log(x-z) + \log(y-z)$

$\therefore \frac{1}{f}\cdot f_x = \frac{1}{x-y} + \frac{1}{x-z}$ ——— (1)

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Similarly, $\frac{1}{f}\cdot f_y = -\frac{1}{x-y} + \frac{1}{y-z}$ ——(2)

$\frac{1}{f}\cdot f_z = -\frac{1}{x-z} - \frac{1}{y-z}$ ——(3)

Adding (1), (2) and (3) we get,

$$\frac{1}{f}(f_x + f_y + f_z) = 0$$

i.e. $f_x + f_y + f_z = 0$. (Proved).

Ex 3. If $u = \log(x^3+y^3+z^3-3xyz)$ then show that

(i) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$

(ii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{3}{(x+y+z)^2}$

(iii) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$

Solⁿ: Note that,

$$x^3+y^3+z^3-3xyz = (x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$$

where ω is a root of the equation $x^3=1$.

Now, $u = \log(x^3+y^3+z^3-3xyz)$

$= \log(x+y+z) + \log(x+\omega y+\omega^2 z) + \log(x+\omega^2 y+\omega z)$

$\therefore u_x = \frac{1}{x+y+z} + \frac{1}{x+\omega y+\omega^2 z} + \frac{1}{x+\omega^2 y+\omega z}$

$u_y = \frac{1}{x+y+z} + \frac{\omega}{x+\omega y+\omega^2 z} + \frac{\omega^2}{x+\omega^2 y+\omega z}$

$u_z = \frac{1}{x+y+z} + \frac{\omega^2}{x+\omega y+\omega^2 z} + \frac{\omega}{x+\omega^2 y+\omega z}$

$\therefore u_x + u_y + u_z = \frac{3}{x+y+z}$, since $1+\omega+\omega^2 = 0$.

which prove (i).

Now, $u_{xx} = \frac{-1}{(x+y+z)^2} + \frac{-1}{(x+\omega y+\omega^2 z)^2} + \frac{-1}{(x+\omega^2 y+\omega z)^2}$

$u_{yy} = \frac{-1}{(x+y+z)^2} + \frac{-\omega^2}{(x+\omega y+\omega^2 z)^2} + \frac{-\omega^4}{(x+\omega^2 y+\omega z)^2}$

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$$u_{zz} = \frac{-1}{(x+y+z)^2} + \frac{-\omega^4}{(x+\omega y+\omega^2 z)^2} + \frac{-\omega^2}{(x+\omega^2 y+\omega z)^2}$$

$$\therefore\ u_{xx} + u_{yy} + u_{zz} = \frac{-3}{(x+y+z)^2}, \text{ since } 1+\omega+\omega^2 = 0 \text{ and } \omega^4 = \omega.$$

which prove (ii).

Also,

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= \frac{-3}{(x+y+z)^2} + \frac{-3}{(x+y+z)^2} + \frac{-3}{(x+y+z)^2}$$

$$= -\frac{9}{(x+y+z)^2}$$

which prove (iii).

Ex.4. Show that if $u(x,y,z)$ satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

then (i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ satisfy it

and also (ii) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$ satisfy it.

Solⁿ:-

Hints:-(i) Find $\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{\partial u}{\partial x}\right)$.

$$= \frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial y^2 \partial x} + \frac{\partial^3 u}{\partial z^2 \partial x}$$

$$= \frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 u}{\partial x \partial z^2}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \frac{\partial}{\partial x}(0) = 0$$

Similarly for $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$.

Hints:-(ii) Take $u_1 = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

Find $\frac{\partial^2 u_1}{\partial x^2}, \frac{\partial^2 u_1}{\partial y^2}, \frac{\partial^2 u_1}{\partial z^2}$ and then add all these.